NONSTATIONARY DISTRIBUTION OF CONCENTRATIONS IN A CHEMOTRON CONVERTER

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The problem of nonstationary distribution of the concentration of an oxidized component of the electrolyte in the cathode region of a chemotron converter is solved.

It is well known that the most important characteristics of a chemotron converter are determined by the distribution of the concentration of the oxidized component of the electrolyte in the cathode region [4].

We consider the problem of computing the time-varying concentration of the oxidized component of the electrolyte in the cathode region of a chemotron converter.

We shall assume that the cathode channel is a rectangular parallelopiped of finite dimensions

$$\Omega : \begin{cases} 0 \leqslant x \leqslant a, \\ 0 \leqslant y \leqslant l, \\ 0 \leqslant z \leqslant h. \end{cases}$$

The surfaces $x = 0$ and $x = a$ of the parallelopiped and the cathode of the converter are made of a material that is chemically inert to the used electrolyte. The walls $z = 0$ and $z = h$ are assumed to be made of a chemically stable insulating material. We assume further that the convection rate is constant and is directed along the 0y axis and that the molecular diffusion along the 0z axis can be neglected, i.e., $\partial^2 C/\partial z^2 = 0$.

Under the condition of stationary flow of the liquid along the channel the equation of convective diffusion has the form [2]

$$\frac{\partial C}{\partial t} + \bar{v}\nabla C = D\nabla^2 C.$$

If we use the above model of the cathode region, then with the assumption made above the problem under investigation reduces to the integration of the differential equation

$$\frac{\partial C}{\partial t} + v \cdot \frac{\partial C}{\partial y} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right) \tag{1}$$

with the boundary conditions

$$C(0,\ y,\ t) = 0, \quad 0 \leqslant y \leqslant l, \quad t > 0, \tag{2}$$

$$C(a,\ y,\ t) = 0, \quad 0 \leqslant y \leqslant l, \quad t > 0, \tag{3}$$

$$C(x,\ 0,\ t) = C_0, \quad 0 < x < a, \quad t > 0, \tag{4}$$

$$C(x,\ l,\ t) = C_1, \quad 0 < x < a, \quad t > 0 \tag{5}$$

and the initial condition

$$C(x,\ y,\ 0) = C_0, \quad 0 \leqslant x \leqslant a, \quad 0 \leqslant y \leqslant l. \tag{6}$$

Here C_0 is the concentration of the oxidized component of the electrolyte at the entrance to the cathode region, and C_1 is the concentration at the exit from it, taken along the direction of flow of the electrolyte.

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We shall seek the solution in the form of the sum

$$C(x, y, t) = \bar{C}(x, y) + \bar{\bar{C}}(x, y, t), \tag{7}$$

where $\bar{C}(x, y)$ is the solution of the stationary problem ($\partial C/\partial t = 0$), satisfying the boundary conditions (2)-(5), and $\bar{\bar{C}}(x, y, t)$ is the solution of the nonstationary problem with initial conditions $C_0 - \bar{C}(x, y)$ and homogeneous boundary conditions, i.e.,

$$\bar{\bar{C}}(0, y, t) = 0, \qquad 0 \leqslant y \leqslant l, \quad t > 0, \tag{2'}$$

$$\bar{\bar{C}}(a, y, t) = 0, \qquad 0 \leqslant y \leqslant l, \quad t > 0, \tag{3'}$$

$$\bar{\bar{C}}(x, 0, t) = 0, \qquad 0 < x < a, \quad t > 0, \tag{4'}$$

$$\bar{\bar{C}}(x, l, t) = 0, \qquad 0 < x < a, \quad t > 0, \tag{5'}$$

$$\bar{\bar{C}}(x, y, 0) = C_0 - \bar{C}(x, y), \quad 0 \leqslant x \leqslant a, \quad 0 \leqslant y \leqslant l. \tag{6'}$$

It is easy to verify that the solution of the stationary problem satisfying boundary conditions (2)-(5) is the function

$$\bar{C}(x, y) = \frac{4}{\pi} \exp\left\{\frac{vy}{2D}\right\} \sum_{k=1}^{\infty} \frac{1}{(2k-1)\operatorname{sh}\nu_{2k-1} l} \left\{ C_1 \exp\left\{-\frac{lv}{2D}\right\} \operatorname{sh}\nu_{2k-1} y + C_0 \operatorname{sh}\nu_{2k-1}(l-y)\right\} \sin\frac{(2k-1)\pi}{a} x. \tag{8}$$

Here

$$\nu_{2k-1} = \sqrt{\left(\frac{v}{2D}\right)^2 + \left(\frac{2k-1}{a}\pi\right)^2}. \tag{9}$$

Further, it can be shown that the function

$$\bar{\bar{C}}(x, y, t) = \exp\left\{\frac{vy}{2D}\right\} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} M_{n,m} \exp\{-\lambda_{n,m}^2\} \sin\frac{n\pi}{a} x \sin\frac{m\pi}{l} y, \tag{10}$$

where

$$\lambda_{n,m}^2 = D\left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{l}\right)^2\right] + \frac{v^2}{4D},$$

$$M_{n,m} = \frac{4}{al} \int_0^a\int_0 [C_0 - \bar{C}(x, y)] \exp\left\{-\frac{vy}{2D}\right\} \sin\frac{n\pi}{a} x \sin\frac{m\pi}{l} y\,dx\,dy, \tag{11}$$

satisfies Eq. (1) and conditions (2')-(6').

Thus the complete solution of the investigated problem is the sum of the functions (8) and (10).

Formulas (8), (10), and (11) are considerably simplified if it is assumed that at the exit from the cathode channel a constant concentration $C_1 = 0$ is maintained. Then the stationary distribution of the concentration of the oxidized component in the cathode region will be determined by the function

$$\tilde{C}(x, y) = \frac{4}{\pi} \exp\left\{\frac{vy}{2D}\right\} \sum_{k=1}^{\infty} \frac{C_0 \operatorname{sh}\nu_{2k-1}(l-y)}{(2k-1)\operatorname{sh}\nu_{2k-1} l} \sin\frac{(2k-1)\pi}{a} x. \tag{12}$$

Series (12) converges quite rapidly everywhere if y does not tend to zero. It can be written in the form of two series in the following way:

$$\tilde{C}(x, y) = \frac{4C_0}{\pi} \exp\left\{\frac{vy}{2D}\right\} \left\{ \sum_{k=1}^{\infty} \frac{\exp\{-\nu_{2k-1} y\}}{2k-1} \sin\frac{(2k-1)\pi}{a} x - \sum_{k=1}^{\infty} \frac{1}{2k-1} \frac{\operatorname{sh}\nu_{2k-1} y}{\operatorname{sh}\nu_{2k-1} l} \exp\{-\nu_{2k-1} l\} \sin\frac{(2k-1)\pi}{a} x \right\}.$$

For all real converter systems v and a are appreciably smaller than unity; therefore, as seen from (9), we can take

$$\nu_{2k-1} = \frac{(2k-1)\pi}{a}.$$

with a large degree of accuracy.

Making this substitution in the first of the series we have

$$\tilde{C}(x, y) = \frac{4C_0}{\pi} \exp\left\{\frac{vy}{2D}\right\}\left\{\sum_{k=1}^{\infty} \frac{1}{2k-1} \operatorname{Im}\left[\exp\left\{i\frac{(2k-1)\pi z}{a}\right\}\right] - \sum_{k=1}^{\infty} \frac{1}{2k-1}\cdot\frac{\operatorname{sh}\nu_{2k-1}y}{\operatorname{sh}\nu_{2k-1}l}\exp\{-\nu_{2k-1}l\}\sin\frac{(2k-1)\pi x}{a}\right\},$$

where z = x + iy.

But it is known [1] that

$$\sum_{k=1}^{\infty} \frac{1}{2k-1} \exp\left\{i\frac{(2k-1)\pi z}{a}\right\} = \operatorname{Arth}\left[\exp\left\{\frac{i\pi z}{a}\right\}\right].$$

Therefore

$$\tilde{C}(x, y) = \frac{4C_0}{\pi} \exp\left\{\frac{vy}{2D}\right\}\left\{\operatorname{Im}\left[\operatorname{Arth}\left(\exp\left\{\frac{i\pi z}{a}\right\}\right)\right] - \sum_{k=1}^{\infty} \frac{\exp\{-\nu_{2k-1}l\}}{2k-1}\,\frac{\operatorname{sh}\nu_{2k-1}y}{\operatorname{sh}\nu_{2k-1}l}\sin\frac{(2k-1)\pi}{a}x\right\},$$

or [1]

$$\tilde{C}(x, y) = \frac{2C_0}{\pi} \exp\left\{\frac{vy}{2D}\right\}\left\{\operatorname{arctg}\left[\frac{\sin\frac{\pi x}{a}}{\operatorname{sh}\frac{\pi y}{l}}\right] - \sum_{k=1}^{\infty} \frac{\exp\{-\nu_{2k-1}l\}}{2k-1}\,\frac{\operatorname{sh}\nu_{2k-1}y}{\operatorname{sh}\nu_{2k-1}l}\sin\frac{(2k-1)\pi}{a}x\right\}.$$

The sum of the last series is small everywhere in the open rectangle $0 \le x \le a$, $0 < y \le l$, and the series converges very rapidly [3]. Therefore, for engineering computations only the first term of the series need be used and we can take

$$\tilde{C}(x, y) = \frac{2C_0}{\pi} \exp\left\{\frac{vy}{2D}\right\}\left\{\operatorname{arctg}\left[\frac{\sin\frac{\pi x}{a}}{\operatorname{sh}\frac{\pi y}{l}}\right] - 2\exp\{\nu_1 l\}\frac{\operatorname{sh}\nu_1 y}{\operatorname{sh}\nu_1 l}\sin\frac{\pi x}{a}\right\} \tag{13}$$

with a large degree of accuracy. In the abovementioned case, expression (13) can be used in formulas (7), (10), and (11) instead of expression (8).

NOTATION

- C is the concentration;
- $\bar{v}$ is the rate of convection of the electrolyte;
- D is the diffusion coefficient;
- t is the time;
- ∇ is the Hamilton operator;
- Ω is the rectangular parallelopiped.

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